Signals and Systems

Lecture 13:

Sampling and Aliasing

Today's lecture

– Aliasing
– Folding
– Ideal Reconstruction
– D-to-A Reconstruction
– Pulse Shapes for Reconstruction
– Sampling Theorem & Band-limited Signals

The Rest of The Story

- Spectrum of *x[n]* has more than one line for each complex exponential:
  - Called **aliasing,**
  - **Many spectral lines.**

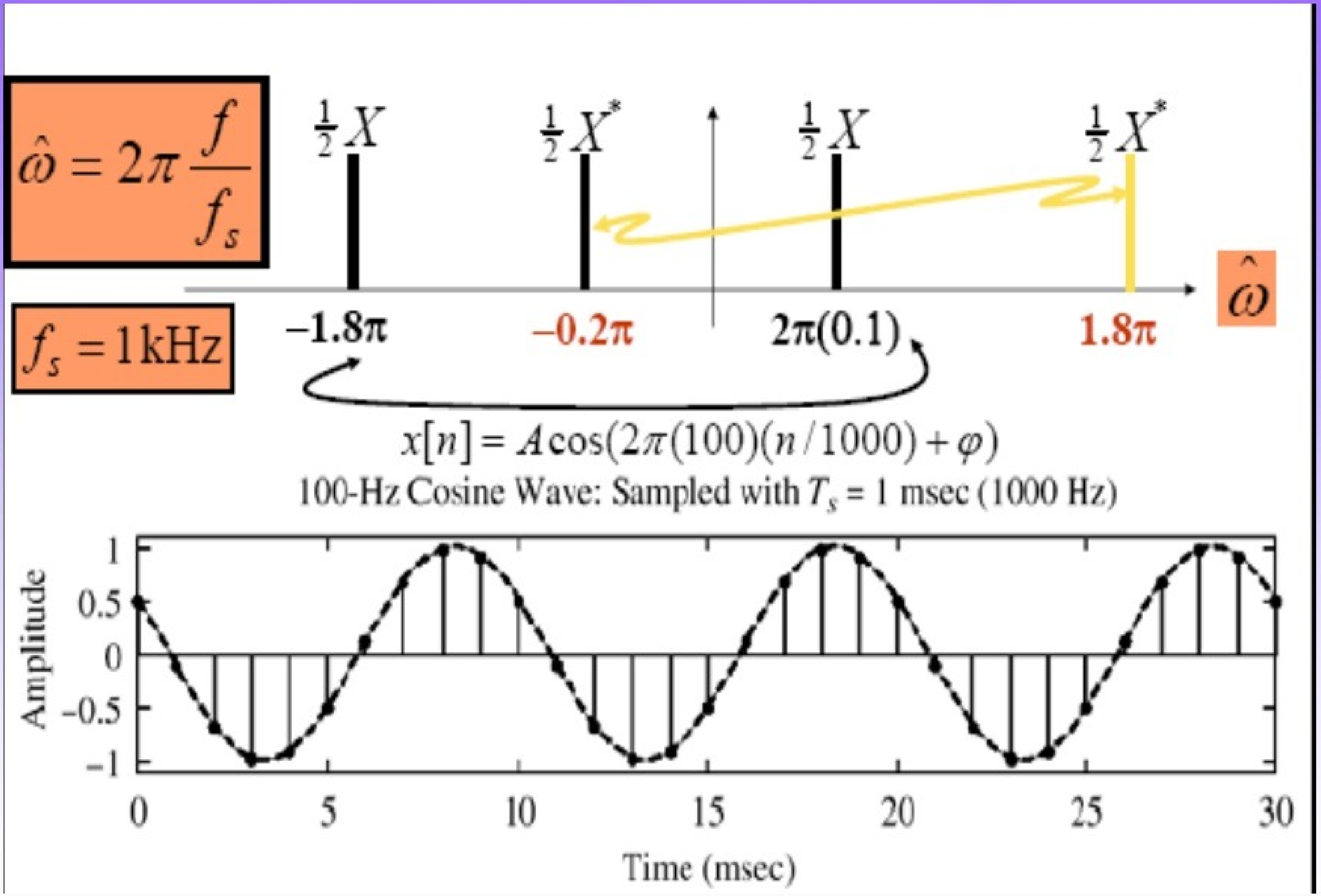
- Spectrum of *x[n]* is periodic with period = **2π** (for angular discrete frequencies), because

$$A\cos(\hat{\omega}n+\varphi)=A\cos((\hat{\omega}+2\pi)n+\varphi)$$

Spectrum for *x[n]*

- Plot versus discrete (normalized) frequency.
- Include **All** Spectral Lines
  - Aliases for angular digital (normalized) frequencies:
    - add multiples of 2π
    - subtract multiples of 2π.
  - Folded Aliases
    - (to be discussed later)
    - aliases of negative frequencies.

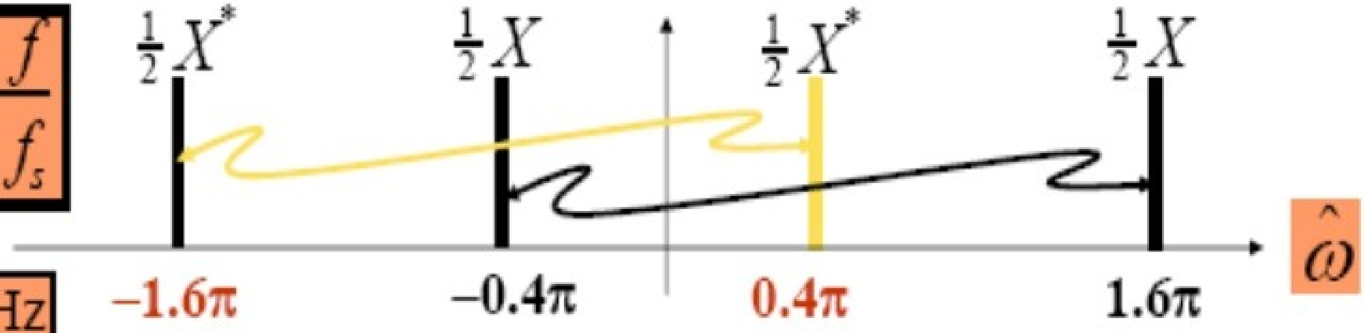
Spectrum (More Lines)



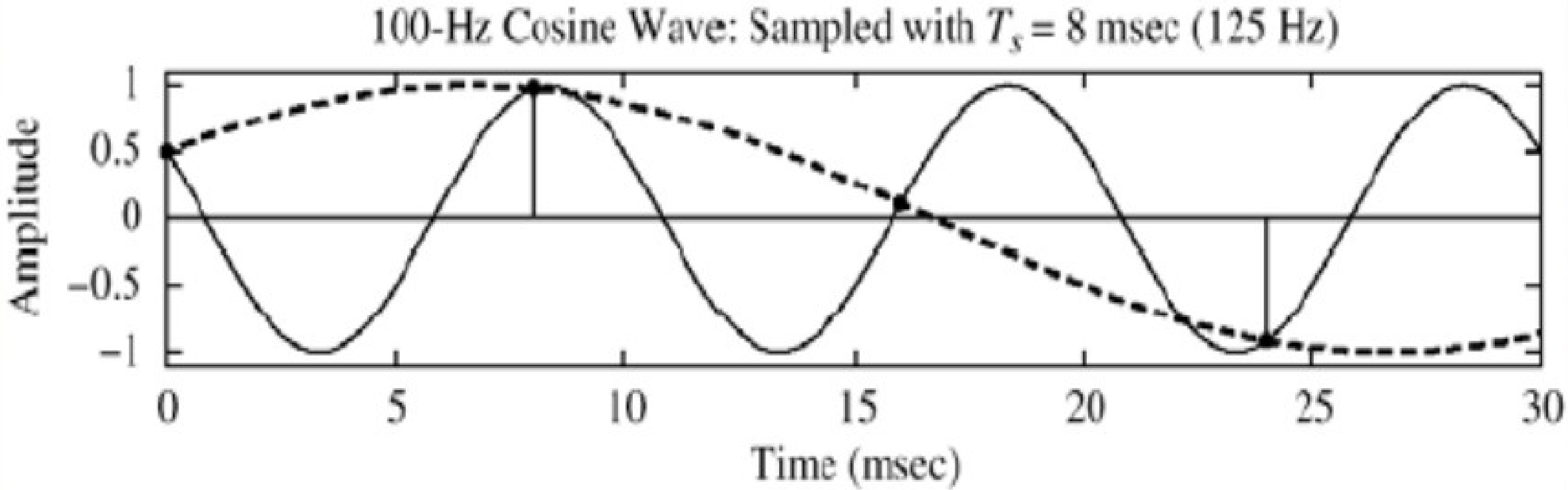
Spectrum (Folding Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 125\text{Hz}$$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



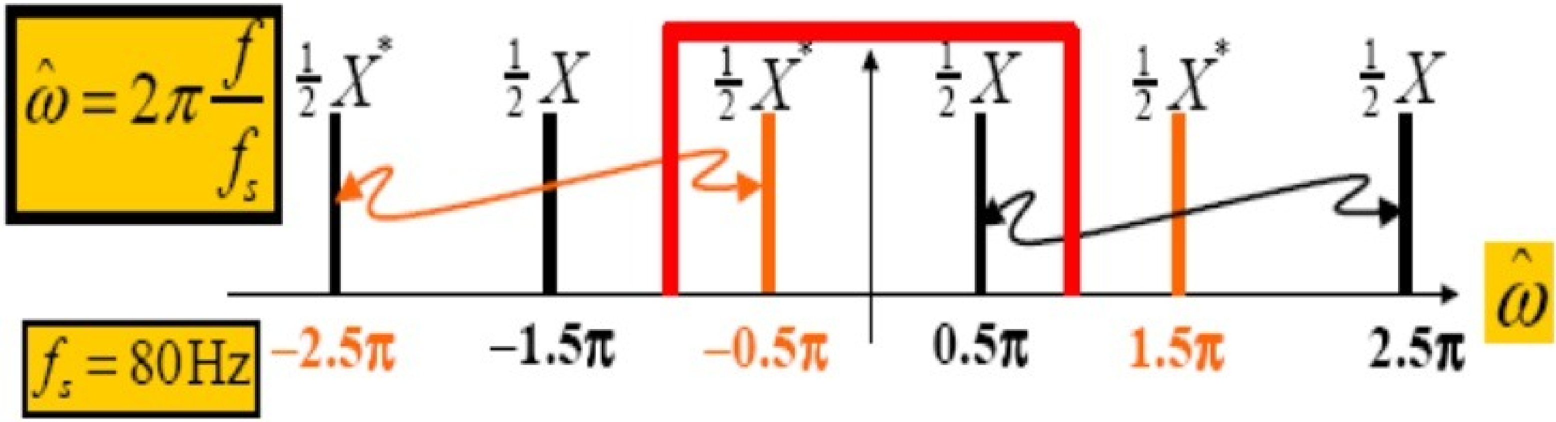
D-to-A Reconstruction

- Create continuous y(t) from y[n]
  - **IDEAL**
    - If you have formula for y[n]
  - Replace n in y[n] with $f_s t$
  - $y[n] = A\cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
  - $y(t) = A\cos(2\pi(800)t + \phi)$

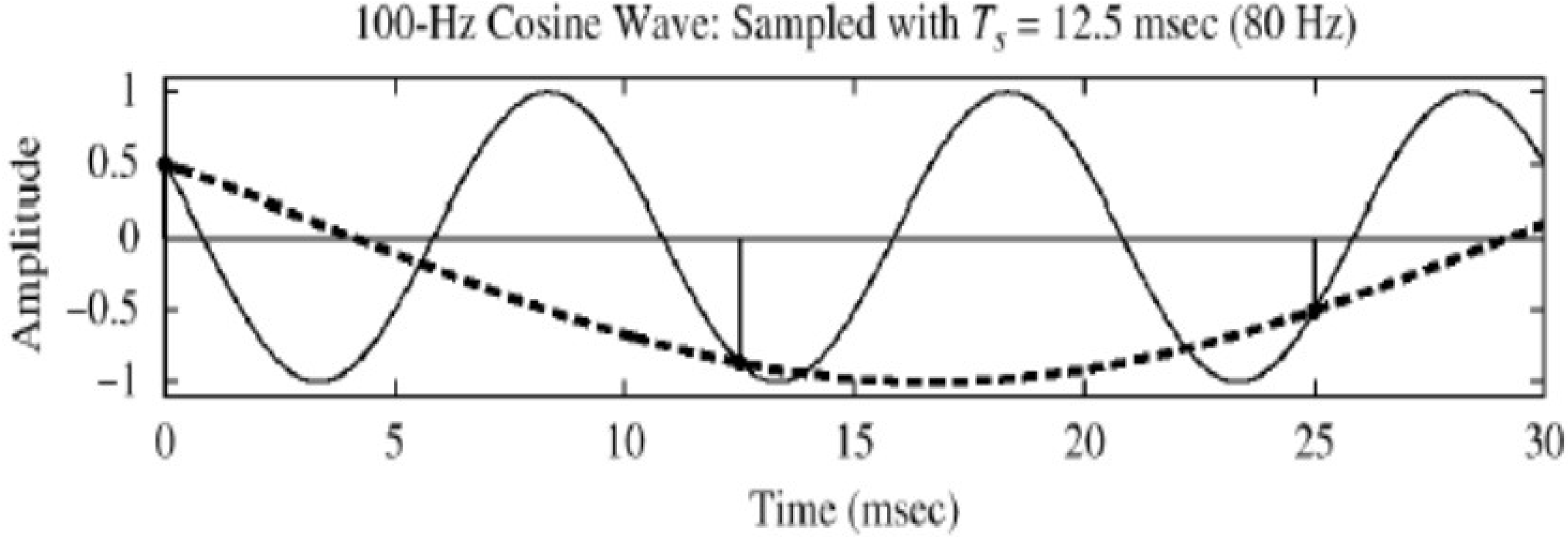
D-to-A is Ambiguous!

- ALIASING
  - <u>Given y[n], which y(t) do we pick ? ? ?</u>
  - INFINITE NUMBER of y(t)
    - PASSING THRU THE SAMPLES, y[n]
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT
- RECONSTRUCT THE <u>SMOOTHEST</u> ONE
  - THE LOWEST FREQ, if y[n] = sinusoid

Spectrum (Aliasing Case)

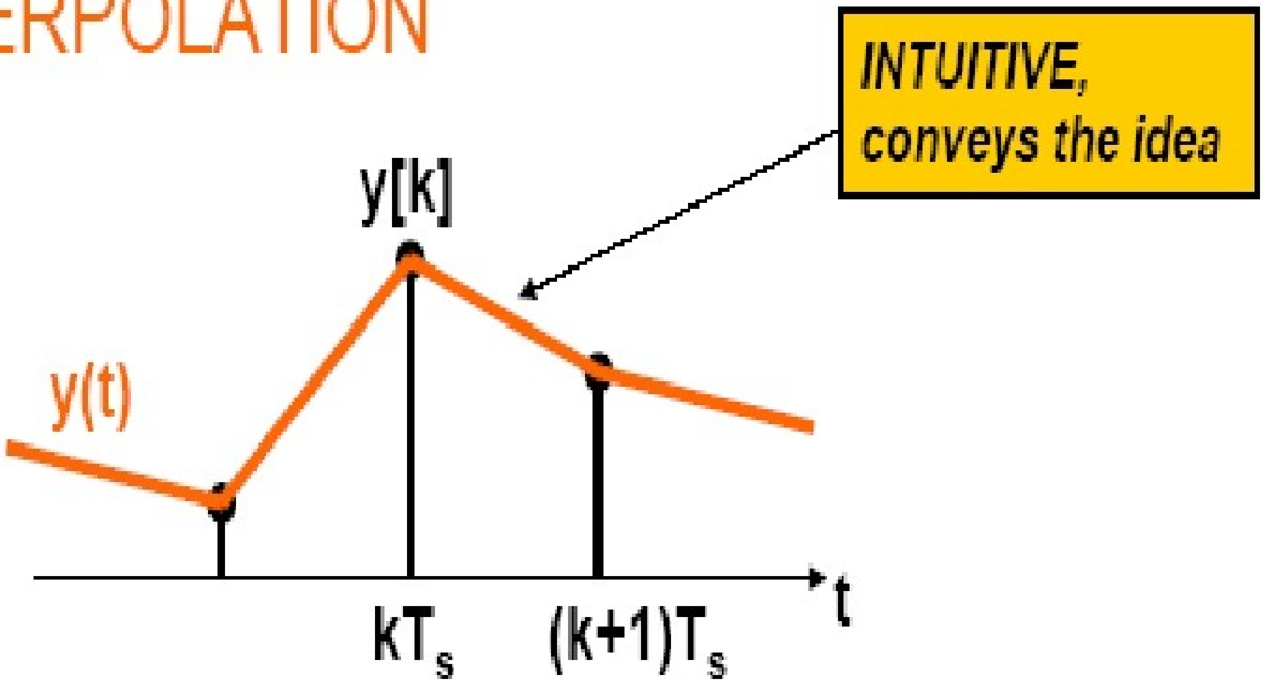


$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$



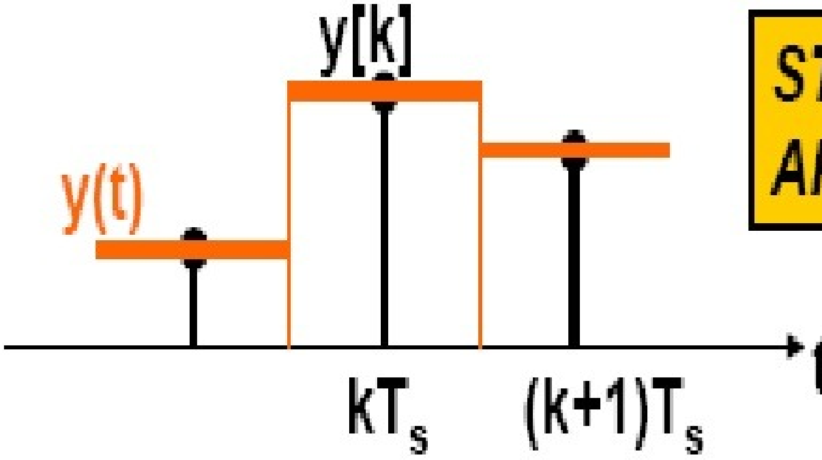
Reconstruction (D-to-A)

- CONVERT STREAM of NUMBERS to x(t)
- "CONNECT THE DOTS"
- INTERPOLATION



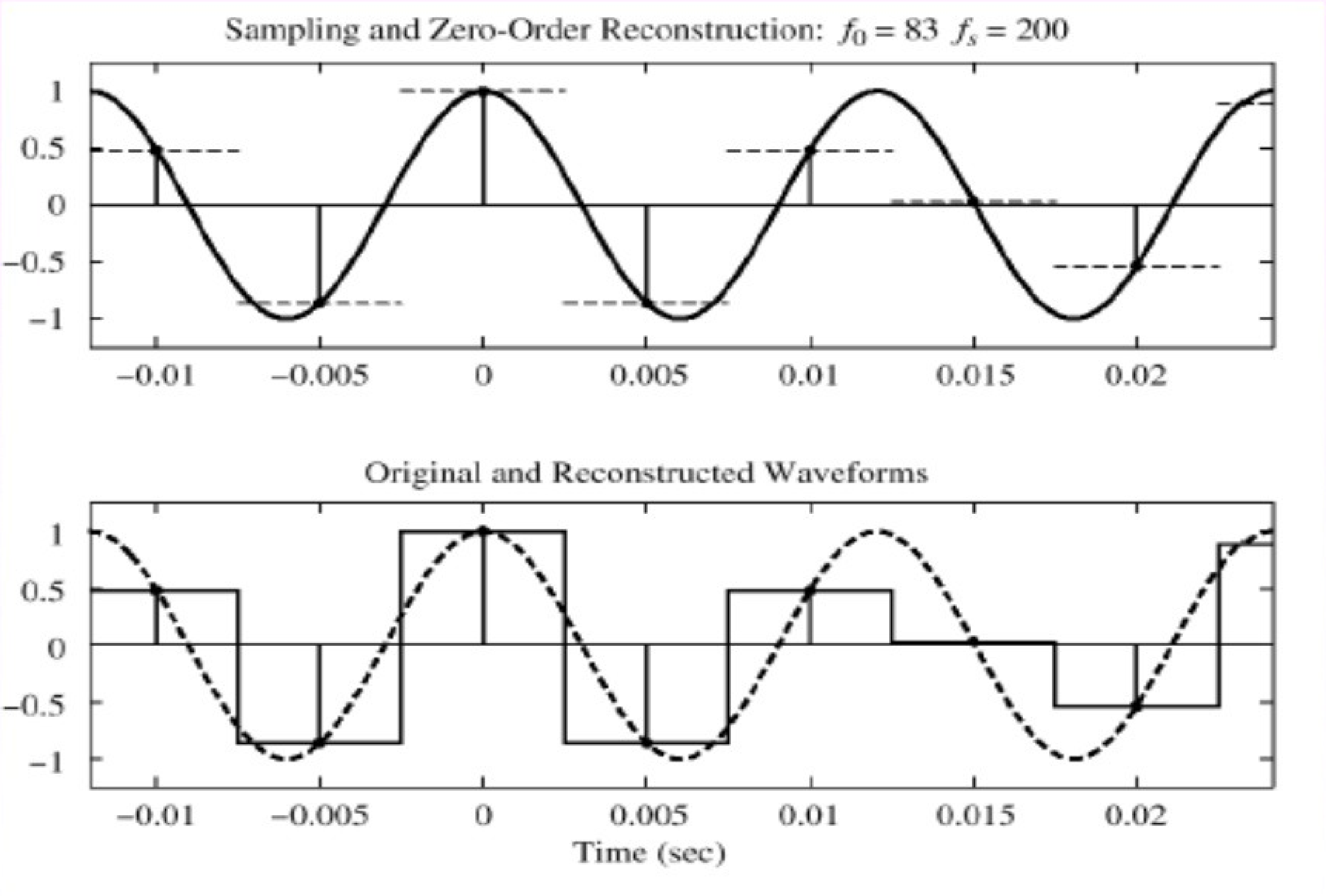
Sample & Hold Device

- CONVERT y[n] to y(t)
  - y[k] should be the value of y(t) at $t = kT_s$
  - Make y(t) equal to y[k] for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$

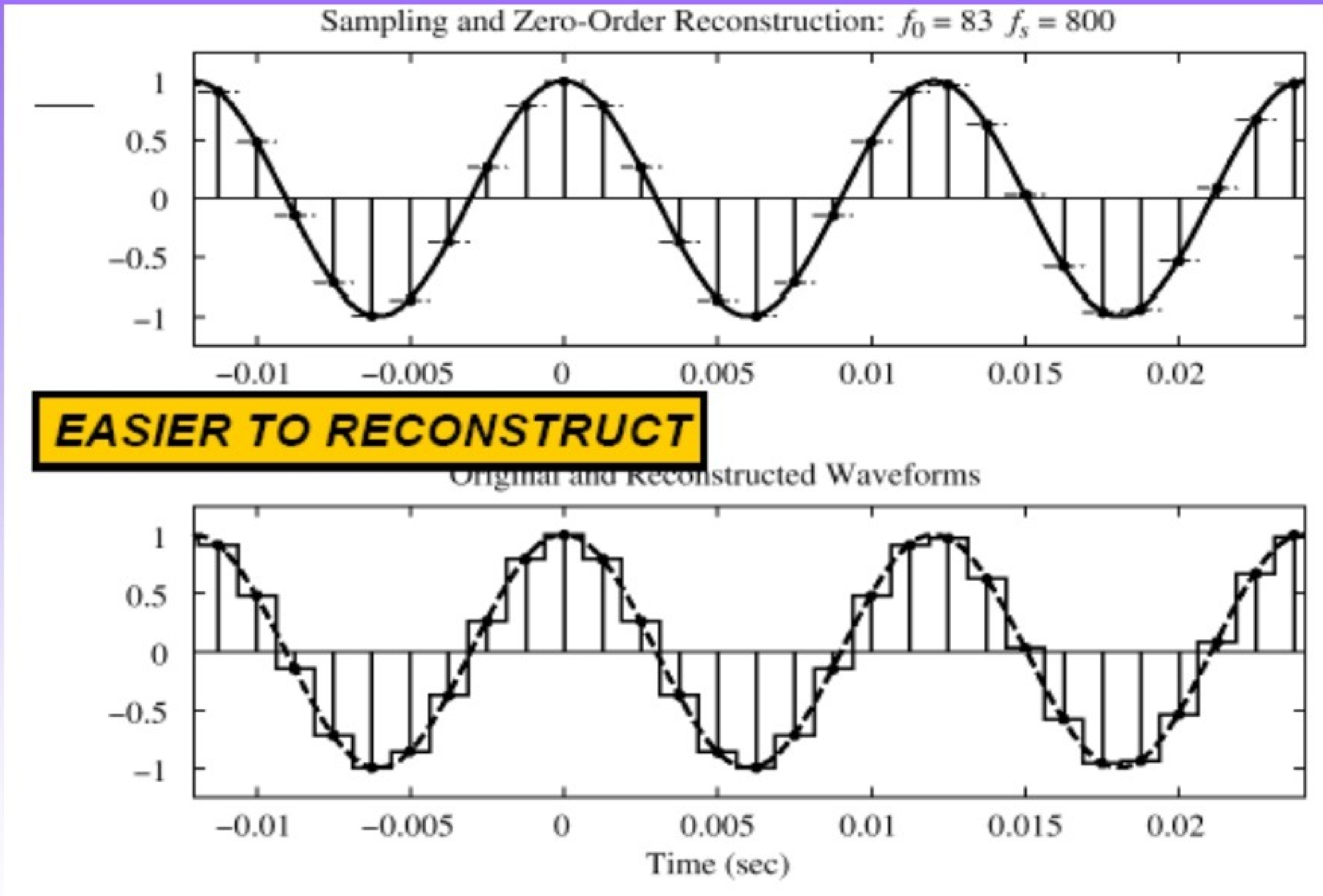


STAIR-STEP APPROXIMATION

Square Pulse Case



Over-Sampling Case



Math Model for D-to-A

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

SQUARE PULSE:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

Expand the Summation

$$\sum_{n=-\infty}^{\infty} y[n]p(t-nT_s) =$$

$$\ldots + y[0]p(t) + y[1]p(t-T_s) + y[2]p(t-2T_s) + \ldots$$

- SUM of SHIFTED PULSES $p(t\text{-}nT_s)$
  - "WEIGHTED" by y[n]
  - CENTERED at $t=nT_s$
  - SPACED by T_s
    - RESTORES "REAL TIME"

Pulse Shapes

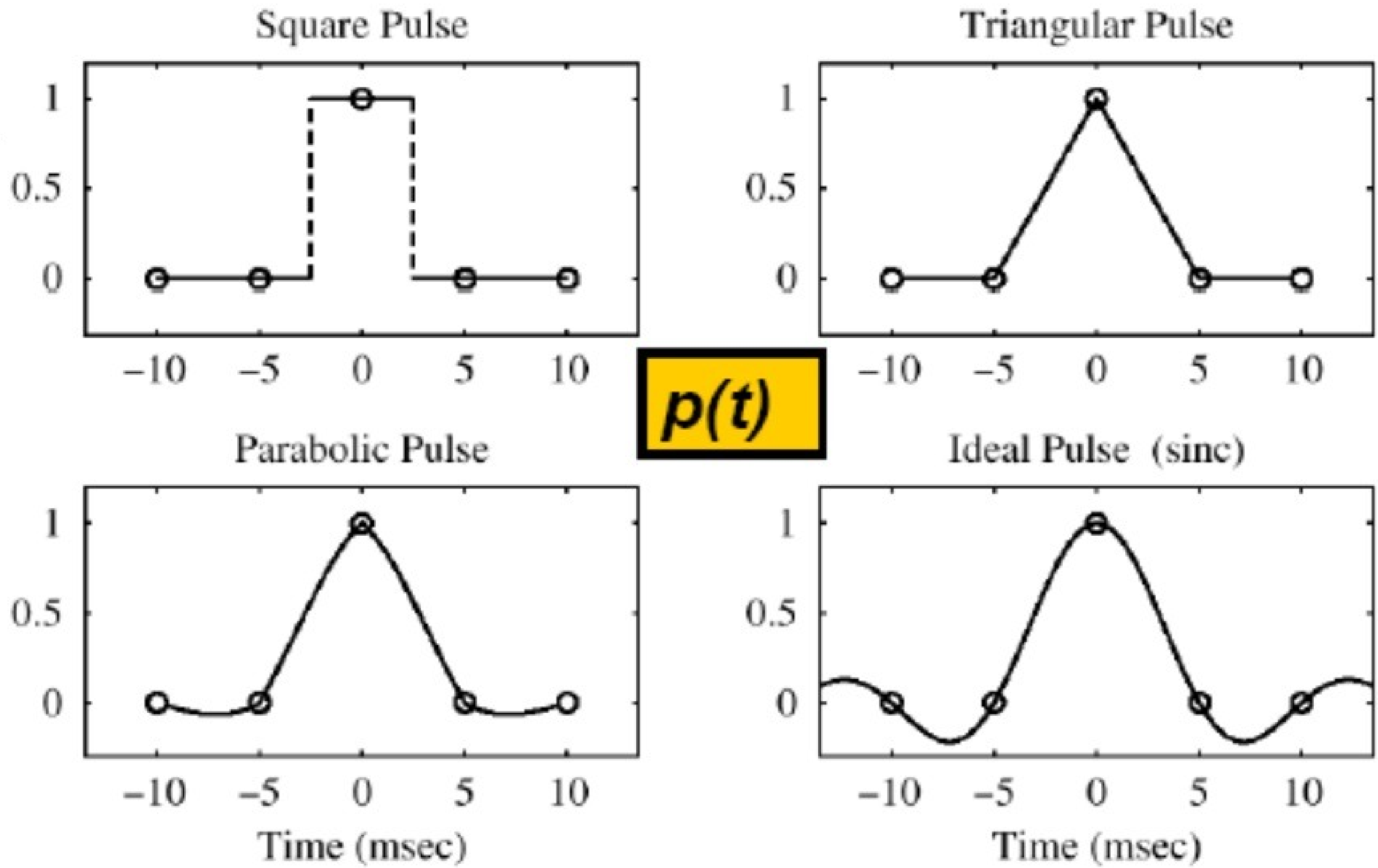
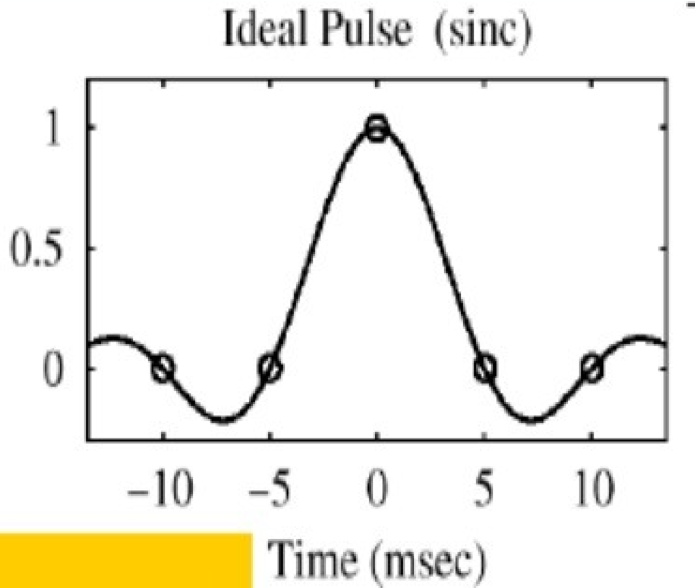


Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times T_s.

Optimal Pulse

CALLED "BANDLIMITED INTERPOLATION"



$$p(t) = \frac{\sin\frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} \quad \text{for} -\infty < t < \infty$$

$$p(t) = 0 \quad \text{for } t = \pm T_s, \pm 2T_s, \ldots$$